The κ-Minkowski Spacetime: Trace, Classical Limit and Uncertainty Relations

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Abstract

Starting from a discussion of the concrete representations of the coordinates of the κ-Minkowski spacetime (in 1+1 dimensions, for simplicity), we explicitly compute the associated Weyl operators as functions of a pair of Schrödinger operators. This allows for explicitly computing the trace of a quantised function of spacetime. Moreover, we show that in the classical (i.e. large scale) limit the origin of space is a topologically isolated point, so that the resulting classical spacetime is disconnected. Finally we show that there exist states with arbitrarily sharp simultaneous localisation in all the coordinates; in other words, an arbitrarily high energy density can be transferred to spacetime by means of localisation alone, which amounts to say that the model is not stable under localisation.

1 Introduction

The κ-Minkowski commutation relations are [1, 2, 3]

$$[q^0, q^j] = \frac{i}{\kappa} q^j, \quad [q^j, q^k] = 0, \quad q^{0*} = q^0, \quad q^{j*} = q^j,$$

where $k, j = 1, \ldots, d$. Usually the real parameter κ is taken of order of a Planck mass; here we will set $\kappa = 1$ (natural units). For simplicity we specialise to the case $d = 1$; defining $T = q^0$, $X = q^1$, these commutation relations become

$$[T, X] = iX. \tag{1}$$

However, our remarks take over to the general case, including the physically relevant $d = 3$; in this short note we outline some of the results of a forthcoming paper [4].

We will begin by fixing an appropriate definition of regular representations, which amounts to formulate them in the stronger form of commutation relations between the unitary one parameter groups generated by X and T. Accordingly,

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we will describe all the irreducible regular representations (see also [9]). We will observe that, since the spectrum $\mathbb{R}$ of the most general regular position operator X is singular in the origin for any value of κ, the origin will remain an isolated point also in the classical limit. In other words, in the classical limit of the κ-Minkowski spacetime it will not be allowed to continuously travel from one side to the other of the origin: an impenetrable barrier will cut the limiting classical spacetime in two decoupled halves.

We then will compute explicitly the Weyl operator $W(\alpha, \beta) = e^{i(\alpha T + \beta X)}$. The composition rule of Weyl operators which results from the regular commutation relations matches those obtained by integrating the Baker–Campbell–Hausdorff (BCH) formula [5], thus justifying a posteriori the application of abstract Lie algebraic methods to the calculus with the Weyl operators. Correspondingly, we will obtain a well defined star product associated with the Weyl calculus (see also [6, 7]). In addition, the explicit knowledge of the Weyl operators will enable us to explicitly compute the trace of the associated Weyl quantisation of a classical function. In the appendix, we briefly recall the relationship between commutation relations of operators and Lie type relations, in order to ease the comparison with the existing literature.

We will complement our discussion by describing how to provide states saturating the Heisenberg uncertainty relations implied by the commutation relations. Indeed, Heisenberg theorem only provides us with lower bounds.

At the end we draw some conclusions.

2   Representations and Classical Limit

The relations (1) are not sufficient to fix a unique model, and we need a regular form (see the appendix for a reminder of motivations). In order to guess it, we first seek for a nontrivial representation ($X \neq 0$). Using the well known relation $[P, f(Q)] = -if'(Q)$, where $P = -id/ds, Q = s\cdot$ are the usual Schrödinger operators on $L^2(\mathbb{R}, ds)$, we easily find a representation by setting

$$T = P, \quad X = e^{-Q}.$$

By computing the explicit action of the unitary groups $e^{i\lambda T}, e^{i\lambda X}$, we find

$$e^{i\alpha T} e^{i\beta X} = e^{i\beta e^{-\alpha} X} e^{i\alpha T}, \quad \alpha, \beta \in \mathbb{R}, \tag{2}$$

Operators T, X fulfilling the above relations are said—by definition—a regular representation of the relations (1).

It is now immediate to check that the choice $T = P, X = -e^{-Q}$ (note the sign) also fulfils the above relations; indeed it can be shown directly [4] that the two pairs ($T = P, X = \pm e^{-Q}$) are the only irreducible, non trivial representations of the relations (2), up to unitary equivalence. The uniqueness argument relies essentially on that of von Neumann for the Schrödinger operators. Our results are essentially equivalent to those of [9], obtained in a different setting using the theory of induced representations.

The trivial representations[1] are, by definition, those where T is any selfadjoint operator, and $X = 0$.

To sum up, for an irreducible regular representation, there are only three possibilities:

1. X is strictly positive;

2. X is strictly negative;

3. $X = 0$.

By definition, the most general admissible representation T, X of our relations will have to contain all the above mentioned irreducibles, otherwise the position operator X would fail to have the whole $\mathbb{R}$ as its spectrum, and we would not be entitled to call our model a quantisation of $\mathbb{R}^2$. In what follows, we shall term "universal" the representation which contains any irreducible precisely once; it generates the C*-algebra of the commutation relations, which turns out to be $\mathcal{K} \oplus \mathcal{C}_0(\mathbb{R}) \oplus \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on the separable, infinite dimensional Hilbert space.

More precisely, the most general admissible position operator X has spectrum $\sigma(X) = \mathbb{R}$ decomposed as

$$\sigma(X) = \sigma_{\text{sing}}(X) \cup \sigma_{\text{cont}}(X),$$

where the singular and continuous spectra are

$$\sigma_{\text{sing}}(X) = \{0\}, \quad \sigma_{\text{cont}}(X) = \mathbb{R} - \{0\}.$$

Now, it is remarkable that the above does not depend on the value of κ (here set equal to one), hence it is bound to survive the classical limit. This means that, as $\kappa \to \infty$, X will go to a continuous function (the usual coordinate function x) of $\mathbb{R}$, which will have 0 as an isolated point of its range[2]. For this not to be in conflict with the asserted continuity, $\mathbb{R}$ must come equipped with an unusual topology which makes 0 an isolated point. Thus, the classical limit of the two dimensional κ-Minkowski spacetime is $\mathbb{R} \times \mathbb{R}$ as a set; but as a topological space, it equals $\mathbb{R} \times \tilde{\mathbb{R}}$ where

$$\tilde{\mathbb{R}} = (-\infty, 0) \sqcup \{0\} \sqcup (0, \infty)$$

is the topologically disjoint union of the two open half lines and the origin.

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[1] by Schur's lemma, the irreducible trivial representations act on the one dimensional Hilbert space $\mathbb{C}$, and $T = c\cdot$ is a multiple of the identity with $c \in \mathbb{R}$; standard direct integral techniques yield a highly reducible trivial representation which contains all the trivial ones precisely once: the latter can be equivalently obtained by setting $T = Q, X = 0$ on $L^2(\mathbb{R})$.

[2] Note that, in an algebra of continuous functions, the range is the same as the spectrum.

3 Weyl Operators and Quantisation

A direct, explicit computation of the Weyl operators $W(\alpha, \beta) = e^{i(\alpha T + \beta X)}$ is not an easy task. It is much easier to guess them, and check the guess a posteriori by means of the Stone–von Neumann theorem. To this end, we remark that the operators we seek for should fulfil three evident requests:

$$W(\alpha, 0) = e^{i\alpha T}, \quad W(0, \beta) = e^{i\beta X},$$
$$W(\alpha, \beta)^{-1} = W(\alpha, \beta)^*,$$
$$W(\lambda\alpha, \lambda\beta)W(\lambda'\alpha, \lambda'\beta) = W((\lambda + \lambda')\alpha, (\lambda + \lambda')\beta)$$

identically for $\alpha, \beta, \lambda, \lambda' \in \mathbb{R}$. The last requirement expresses the remark that, for each α, β fixed, the operator $\alpha T + \beta X$ is selfadjoint, so that $\lambda \mapsto W(\lambda\alpha, \lambda\beta) = e^{i\lambda(\alpha T + \beta X)}$ is a unitary one parameter group. With the ansatz $W(\alpha, \beta) = e^{ir(\alpha,\beta)T}e^{is(\alpha,\beta)X}$, some little effort [4] leads to the solution

$$W(\alpha, \beta) = e^{i\alpha T}e^{i\frac{e^\alpha - 1}{\alpha}\beta X}.$$

In particular if $T = P, X = \pm e^{-Q}$, then we have

$$(W(\alpha, \beta)\xi)(s) = (e^{i\alpha P \pm \beta e^{-Q}}\xi)(s) = e^{\pm\frac{1 - e^{-\alpha}}{\alpha}\beta e^{-s}}\xi(s + \alpha), \quad \xi \in L^2(\mathbb{R}). \quad (3)$$

It is now a routine check to see that, due to the commutation relations, the product of two Weyl operators is again a Weyl operator:

$$W(\alpha_1, \beta_2)W(\alpha_2, \beta_2) = W(\alpha, \beta),$$

where

$$(\alpha, \beta) = (\alpha_1 + \alpha_2, w(\alpha_1 + \alpha_2, \alpha_1)e^{\alpha_2}\beta_1 + w(\alpha_1 + \alpha_2, \alpha_2)\beta_2)$$

is defined in terms of the function

$$w(\alpha, \alpha') = \frac{\alpha(e^{\alpha'} - 1)}{\alpha'(e^\alpha - 1)}$$

which is understood to be extended by continuity to the whole $\mathbb{R}^2$.

In other words, the set of Weyl operators is a subgroup of the group of unitary operators. Moreover, since for the universal representation (T, X) the correspondence

$$(\alpha, \beta) \leftrightarrow W(\alpha, \beta)$$

between $\mathbb{R}^2$ and the group of Weyl operators is one to one, we may use it to endow $\mathbb{R}^2$ with a new group structure, with product

$$(\alpha_1, \beta_1)(\alpha_2, \beta_2) = (\alpha_1 + \alpha_2, w(\alpha_1 + \alpha_2, \alpha_1)e^{\alpha_2}\beta_1 + w(\alpha_1 + \alpha_2, \alpha_2)\beta_2)$$

and identity $(0,0)$. We will denote the resulting group by H. This group will play a role analogous to that played by the Heisenberg group in the case of

the canonical commutation relations (CCR) [8]; we refrain from calling it the κ-Heisenberg group, because such terminology already arose in the framework of quantum groups; moreover, H is not a deformation of the Heisenberg group.

The group H is connected and simply connected; hence it is uniquely associated to its Lie algebra $\mathrm{Lie}(H)$ which is precisely the real Lie algebra with generators u, v and relations $[u, v] = -v$ [4].

The natural ansatz for the quantisation is to interpret the Weyl operators as the quantised plane waves; correspondingly, for a generic ordinary function f we set

$$f(T, X) = \int d\alpha d\beta \; \check{f}(\alpha, \beta) W(\alpha, \beta)$$

where

$$\check{f}(\alpha, \beta) = \frac{1}{(2\pi)^2} \int dt dx \; f(t, x) e^{-i(\alpha t + \beta x)}.$$

The star product is then defined by

$$(f \star g)(T, X) = f(T, X) g(T, X),$$

where the operator product is taken on the right hand side. With this position, a formal computation yields

$$(f \star g)(t, x) =$$
$$= \frac{1}{(2\pi)^2} \int d\alpha \; d\beta \; dy \; dz \; e^{i(\alpha t - \beta y - \alpha z + \beta z)} f(y, w(\alpha, \beta) e^{\alpha - \beta} x) g(z, w(\alpha, \alpha - \beta) x).$$

Indeed, it is rather difficult to directly check that these definitions are well posed for a sufficiently rich class of functions, closed under this product. The reason for this is that, contrary to the case of the Heisenberg group, our H is not unimodular. However there is a way out, which will be described in detail in [4].

4 The Trace

We now will be rewarded of the effort we spent in carrying on explicit computations: we will classify the functions f whose quantisations $f(T, X)$ are trace class as operators on a Hilbert space, and compute explicitly the trace of $f(T, X)$ by means of a functional evaluated on the function f, when T, X is the universal representation.

Let us first fix the representation $(T_+ = P, X_+ = e^{-Q})$ with positive X; given a function f, we seek for a function g such that

$$f(T_+, X_+) = g(P, Q), \tag{4}$$

where for the right hand side we take the canonical Weyl quantisation for the Schrödinger particle on the line. Then we use the well known fact that

$$\mathrm{Tr}(g(P, Q)) = \int dt dx \; g(t, x) =: \tau_+(f).$$

Analogously, fixing the representation with negative X, we define τ_-. For admissible f's, the functionals $\tau_\pm(f)$ only will depend on the values f takes on the half lines $\pm(0, \infty)$, respectively. Finally, for the universal representation we find

$$\mathrm{Tr} f(T, X) = \tau_-(f) + \tau_+(f),$$

where

$$\tau_\pm(f) = \int dt\, dx f(t, \pm e^{-x}),$$

and $f(T, X)$ is trace class if and only if both integrals exists (so that $f(t, 0) = 0$ is a necessary, yet not sufficient condition for $f(T, X)$ to be trace class).

To complete the above discussion, we have to show how to determine the functions g. Here the idea is to compare the integral kernels $K_\pm^f$ of $f(T = P, X = \pm e^{-Q})$ and H^g of $g(P, Q)$, specified here below: for $\xi \in L^2(\mathbb{R})$,

$$(f(P, \pm e^{-Q})\xi)(s) = \int dr\, K_\pm^f(s, r)\xi(r),$$

$$(g(P, Q)\xi)(s) = \int dr\, H^g(s, r)\xi(r),$$

where

$$K_\pm^f(s, r) = \frac{1}{2\pi} \int dt\, f\left(t, \pm\frac{e^{-s} - e^{-r}}{r - s}\right) e^{i(r-s)t},$$

$$H^g(s, r) = \frac{1}{2\pi} \int dt\, g\left(t, \frac{r + s}{2}\right) e^{i(s-r)t}.$$

The kernel H^g is well known from canonical (CCR) Weyl quantisation; $K_\pm^f$ can be directly computed using the explicit action (3) The condition (4) then becomes

$$H^g \equiv K_\pm^f,$$

which has solution

$$g_\pm(t, x) = \int d\alpha\, e^{i\alpha t} f^{\check{}\otimes\mathrm{id}}\left(\alpha, \pm\frac{e^{\alpha/2} - e^{-\alpha/2}}{\alpha} e^{-x}\right).$$

We refer the interested reader to [4] for a more detailed discussion.

5 Uncertainty Relations

Of course, in any trivial representation T, X commute, hence Heisenberg uncertainty has empty content, and simultaneous sharp localisation can be obtained both in space and time with states relative to a trivial representation. This might seem specifically related to the special status of 0 in the spectrum of the most general (universal) representation; that however is not the case, indeed. Fix for example the irreducible representation with positive X, and observe

that, due to the form $X = e^{-Q}$ of the position, a state ξ is localised close to 0 if, as an L^2 function of s, it is essentially supported at large positive s. In other words, the behaviour of ξ at large (small) s is related with localisation of X at small (resp. large) spectral values of $X = e^{-Q}$. Hence one may take a state with any desired uncertainty ε in $T = P$; such a state can be chosen with finite (though large) support as a function of s. By shifting it (as a function of s) on the right sufficiently far from $s = 0$, one may obtain a state sharply localised around the spectral value 0 of $X = e^{-Q}$ with any given uncertainty $\eta > 0$, without affecting the uncertainty ε. Hence the two uncertainties ε and η can be chosen independently, and small at wish.

In conclusion, it is possible to simultaneously localise in time and space, at the only cost of confining the state sufficiently close to the origin. One might give an intuitive description of this state of affairs by saying that the κ-Minkowski spacetime is classical (at any time) close to the origin of space; while, by similar arguments, one might say that it is increasingly noncommutative far away from the origin of space (e.g. at cosmic distances from the origin). Note that, together with the breakdown of translation covariance (implicit in the commutation relations), this gives the origin of space a very special status.

6 Conclusions

While a thorough mathematical discussion of the representation theory (and thus of the associated C*-algebra) is available, giving a complete symbolic calculus in terms of star products and traces associated to the quantisation à la Weyl, on the other side the physical interpretation of the model exhibit some unpleasant features. In particular, the classical limit, though describing the usual spacetime as a set, appears to be endowed with a pathological topology. Moreover, contrary to any physical expectation, it exhibits very large noncommutative effects at large (e.g. cosmic) distances from a privileged point of the space. These features become even more strikingly unpleasant in higher dimensions [4].

Appendix

We recall here some basic facts about representations of Lie relations by selfadjoint operators on a Hilbert space. Firstly, we wish to fix the correspondence between the abstract real Lie relations and the commutation relations of the associated regular representations, which involve the complex structure. Secondly, we wish to emphasise in general that the real Lie algebra underlying the definition of regular representations of the given commutation relations among Hilbert space operators plays an ancillary rôle.

Let $\mathcal{A}$ be a real Lie algebra with generators $u_1, \ldots, u_n$ and relations

$$[u_j, u_k] = \sum_l c_{jkl} u_l, \tag{5}$$

and consider a representation U (by unitary operators on some Hilbert space $\mathfrak{H}$) of the unique connected, simply connected group G with $\mathrm{Lie}(G) = \mathcal{A}$. If $\mathrm{Exp} : \mathcal{A} \to G$ is the usual Lie exponential map, there are uniquely defined selfadjoint operators $A_1, \ldots, A_n$ on $\mathfrak{H}$, such that $U(\mathrm{Exp}[\lambda u_j]) = e^{i\lambda A_j}$ as unitary one–parameter groups of operators. For every choice of the generators $u_{j_1}, \ldots, u_{j_k}$ there are $\mathcal{A}$-valued functions $\alpha^{(k)}_{j_1,\ldots,j_k}$ defined on some open neighbourhood of the origin of $\mathbb{R}^k$ such that

$$\mathrm{Exp}[\lambda_1 u_{j_1}]\mathrm{Exp}[\lambda_2 u_{j_2}]\cdots\mathrm{Exp}[\lambda_k u_{j_k}] = \mathrm{Exp}\left[\sum_l \alpha^{(k)}_{j_1,\ldots,j_k}(\lambda_1,\ldots,\lambda_k)\right].$$

Correspondingly, there are selfadjoint operator valued functions $R^{(k)}_{j_1,\ldots,j_k}$ such that

$$e^{i\lambda_1 A_{j_1}} e^{i\lambda_2 A_{j_2}} \cdots e^{i\lambda_k A_{j_k}} = e^{iR^{(k)}_{j_1,\ldots,j_k}(\lambda_1,\ldots,\lambda_k)}. \tag{6}$$

Formal computations yield

$$\frac{d^2}{d\lambda\,d\lambda'} e^{i\lambda A_j} e^{i\lambda' A_k} e^{-i\lambda A_j}\bigg|_{\lambda=\lambda'=0} = i[A_j, A_k].$$

Hence, using (6), we get

$$[A_j, A_k] = iC_{jk}, \tag{7}$$

where

$$C_{jk} = -\frac{d^2}{d\lambda\,d\lambda'} R^{(3)}_{j,k,j}(\lambda, \lambda', -\lambda)\bigg|_{\lambda=\lambda'=0}.$$

The commutation relations (6) are usually called a regular (or Weyl) form of the commutation relations (7), relative to the given representation U of G. In order to give them an intrinsic meaning, one has to give a criterion to select G and U. Typically, the fundamental physical relations are those in the ordinary form (7) (to be complemented with the implicit requirement that $A_j = A_j^*, C_{jk} = C_{jk}^*$), which are directly related to physical interpretation through the Heisenberg theorem. The *choice* of the corresponding regular relations (i.e. of G and U) is then a subsequent step which is necessary in order to fix the admissible realisations of the model.

The many technical problems afflicting the above formal derivation of (7) (as well as its interpretation) should not be considered just as "technicalities" of no physical interest: indeed, even when the basic relations (7) are nicely fulfilled on some dense domain, they may belong to the representation of a totally different Lie algebra, namely to totally different Weyl relations. There is a striking example, due to Nelson (unpublished; see [10, VIII.5]), of two operators which are essentially selfadjoint on a common stable dense domain, where they commute; yet the unitary groups they generate do not commute! Hence, it is customary to write (7) as a more appealing shorthand for the corresponding regular form (6), which however should be fixed without ambiguity. Typically, the regular form is understood precisely to be the result of a formal application of the Baker–Campbell-Hausdorff formula, which we recall it cannot be applied in general

to unbounded operators. In a sense, regular representations are precisely those particularly nice representations which match with the formula.

These concepts first arose in the famous analysis of the uniqueness problem of the canonical commutation relations

$$[P, Q] = -iI$$

done by von Neumann, by implementing the ideas of Weyl. Starting from a physically motivated choice of the regular representation [11], von Neumann [8] found a Lie group (the Heisenberg group) reproducing precisely the initial regular representations. The study of regular canonical representations then was reduced precisely to the representation theory of the Heisenberg group.

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